SOME RESULTS ON THE WELL-POSEDNESS OF EULER-VOIGT AND NAVIER-STOKES-VOIGT MODELS.

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Abstract. We consider the Euler-Voigt equations and the Navier-Stokes-Voigt equations, which are obtained by an inviscid α-regularization from the corresponding equations. The main result we show is the structural stability of the system in term of the variations of both viscosity of regularization parameters.

1. Introduction

One of the most challenging problems in scientific computing is that of producing reliable simulations of turbulent flows. Since the work of Kolmogorov in 1941, it is well assessed that there are prohibitive limitations due to the smallest scales which are persistent in flows at very high Reynolds number. The limitations (due to speed but also to memory capacity of the available most powerful computers) in performing direct numerical simulations makes the field particularly challenging. Incompressible fluid with constant density are described by the Navier-Stokes equations

$$\partial_t v + (v \cdot \nabla)\, v - \nu \Delta v + \nabla p = f, \tag{1a}$$

$$\nabla \cdot v = 0, \tag{1b}$$

supplemented with initial and boundary conditions, where $v(t,x) = (v_1, v_2, v_3)$ is the velocity field, $p(t,x)$ denotes the pressure, $f(t,x) = (f_1, f_2, f_3)$ is the external force, and $\nu > 0$ is the kinematic viscosity. In the sequel we will consider mainly the problem in the space periodic setting.

One main idea is that of studying averaged or filtered equations, see the mathematical overview in [7]. Many models for the numerical simulation of the Large Scales (the only ones which are effectively computable and of relevance for the applications) has been proposed. Among these models in the recent years there has been a lot of activity, from both the pure and applied mathematicians, around the so called "α-models." These models are based on a filtering/smoothing obtained with the application of the inverse of the Helmholtz operator

$$\mathrm{I} - \alpha^2 \Delta,$$

and two main questions arise: a) How to describe the (nonlinear) quadratic term $(\mathrm{I} - \alpha^2\Delta)^{-1}(v \cdot \nabla)\, v$ in terms of $(\mathrm{I} - \alpha^2\Delta)^{-1} v$ only (interior closure problem in LES); b) The role of boundary conditions supplementing the Helmholtz operator and the derived model. Roughly speaking, the effect of applying the inverse of the Helmholtz operator is that of getting two more derivatives of the solution under control. We also recall that Leray's [28] approach to construct weak solutions of

Date: January 4, 2011.

the Navier-Stokes equations by smoothing just the convective velocity is based on a very similar idea, with regularization made by convolution. There is a big variety in the family of α-approximations to the Navier-Stokes equations and we recall in alphabetic order, without the claim of being exhaustive, some of the recent publications about the subject [8, 9, 10, 11, 13, 14, 15, 16, 24, 25, 26, 27]. Further details can be found in the introduction of [9, 33, 37]. We observe that essentially all the above methods are based on a sort of regularization by a viscous smoothing of the equations, which is reflected in better analytical properties.

A very promising and new approach, recently introduced by Cao, Lunasin, and Titi [10], is that of the inviscid regularization coming from the Layton-Lewandowski (or simplified Bardina) model, which is a zeroth order deconvolution method, when the viscosity vanishes. Authors in [10] observed that setting the viscosity $\nu = 0$ in that model gives the following (with $u = (\mathrm{I} - \alpha^2\Delta)^{-1}v$) Euler-Voigt model:

$$\partial_t u - \alpha^2 \partial_t \Delta u + (u \cdot \nabla)\, u + \nabla p = f, \tag{2a}$$

$$\nabla \cdot u = 0. \tag{2b}$$

Curios enough, when the viscosity is reintroduced, this turns out to coincide with a model for visco-elastic fluids studied starting from the seventies by Oskolkov [34, 35] and known as the Navier-Stokes-Voigt (sometimes written Voight) model:

$$\partial_t u - \alpha^2 \partial_t \Delta u + (u \cdot \nabla)\, u - \nu \Delta u + \nabla p = f, \tag{3a}$$

$$\nabla \cdot u = 0. \tag{3b}$$

The idea of the inviscid regularization is very interesting for two main reasons: 1) this is not a viscous regularization, and the energy behavior is respected in a more precise way; 2) This regularization does not introduce new boundary conditions, and gives chances to the study of the problem in bounded domain, without the aforementioned usual difficulties coming in theory of Large Eddy Simulations, when employed in presence of solid boundaries. Related ideas, based on time relaxation, has been introduced in [26], while the same inviscid regularization has been also used in the study of water waves [6], while recent applications to the quasi-geostrophic equations are given in [22].

A detailed account of many properties of Voigt equations has been given in [17, 18, 23, 29] and, beside giving a motivation based on the role in scientific computing for the study of the Navier-Stokes and Euler Voigt models, in this paper we treat questions more linked with the general theory of partial differential equations, as explained for instance in [4]. In particular, the main results, which are obtained by using in a simplified setting the techniques introduced by Beirão da Veiga [1, 2, 3] concern the well-posedness of the equations, which is also relevant to understand the stability of the equations with respect to small perturbation of the parameters. The results we prove are not surprising and they are strictly linked with the similar ones recently proved by Linshiz and Titi [29]. Nevertheless our results show how the systems are robust and this gives new support for their employment in numerical computations.

Plan of the paper. In Section 2 we introduce the notation and give some remarks on the existence of smooth solutions. In Section 3 we prove the sharp limits as the regularization parameter α vanishes, while in Section 4 we study both limits as ν and α vanish.

2. Some preliminary results

We introduce now the notation and give some remarks on the existence of solutions to the Voigt models.

2.1. **Notation.** In the sequel we will use the customary Lebesgue L^p and Sobolev spaces $W^{k,p}$ and $H^s := W^{s,2}$, and for simplicity we do not distinguish between scalar and vector valued functions. Since we will mainly work with periodic boundary conditions the spaces are made of periodic functions. In the Hilbertian case $p = 2$ we can easily characterize the divergence-free spaces by using Fourier Series on the 3D torus: Define $\Omega := [0, 2\pi]^3$. We denote by (e_1, e_2, e_3) the orthonormal basis of $\mathbb{R}^3$, and by $x := (x_1, x_2, x_3) \in \mathbb{R}^3$ the standard point in $\mathbb{R}^3$. Let $\mathbb{T}$ be the torus defined by $\mathbb{T} := \mathbb{R}^3/\mathbb{Z}^3$. We use $\|\cdot\|$ to denote the $L^2(\mathbb{T})$ norm and we impose the zero mean condition $\displaystyle\int_\Omega \phi\, dx = 0$ on velocity, pressure and external force. We define, for an exponent $s \geq 0$,

$$H_s := \left\{ w : \mathbb{T} \to \mathbb{R}^3,\ w \in H^s(\mathbb{T})^3, \quad \nabla \cdot w = 0, \quad \int_{\mathbb{T}} w\, dx = 0 \right\},$$

where $H^s(\mathbb{T})^3 := \big[H^s(\mathbb{T})\big]^3$ and if $0 \leq s < 1$ the condition $\nabla \cdot w = 0$ must be understood in a weak sense. For $w \in H_s$, we can expand the velocity field with Fourier series

$$w(x) = \sum_{k \neq 0} \widehat{w}_k e^{ik\cdot x}, \quad \text{where } k \text{ is the wave-number},$$

and the Fourier coefficients are given by $\widehat{w}_k = \frac{1}{|\mathbb{T}|}\int_{\mathbb{T}} w(x) e^{-ik\cdot x} dx$, where $|\mathbb{T}|$ denotes the Lebesgue measure of $\mathbb{T}$. If $k := |k| = \sqrt{|k_1|^2 + |k_2|^2 + |k_3|^2}$, then the H_s norm is defined by

$$\|w\|^2_{H_s} = \sum_{k \neq 0} |k|^{2s} |\widehat{w}_k|^2,$$

where, as above, $\|w\|_{H_0} := \|w\|$. The inner products associated to these norms are

$$(w, v)_{H_s} = \sum_{k \neq 0} |k|^{2s} \widehat{w}_k \cdot \overline{\widehat{v}_k}.$$

We finally characterize $H_s \subset H^s(\mathbb{T})$ as follows:

$$H_s := \Big\{ w = \sum_{k \neq 0} \widehat{w}_k e^{ik\cdot x} : \quad \sum_{k \neq 0} |k|^{2s} |\widehat{w}_k|^2 < \infty,\ k \cdot \widehat{w}_k = 0, \widehat{w}_{-k} = \overline{\widehat{w}_k} \ \Big\}.$$

Moreover, we will use the smoothing operator defined for all $\delta > 0$ and for all functions f periodic and divergence free, as follows: For any $m \geq 0$ and given $f \in H_m$, then the function f_δ defined as

$$f_\delta := \sum_{|k| < 1/\delta} \widehat{f}_k e^{ik\cdot x}, \tag{4}$$

is such that $f_\delta \in H_s$ for all $s \in \mathbb{R}$ (hence it is infinitely differentiable) and $f_\delta \to f$ in H_m as $\delta \to 0$. Other properties of this (truncation in wave-number space) smoothing operator can be easily obtained from the Fourier series characterization and will be recalled in Section 3.

As a final remark on function spaces, we will also use (but only in Section 2.3) the characterization of divergence free subspaces of L^2 and H^1, with vanishing

Dirichlet boundary conditions for which we refer for instance to Constantin and Foias [12].

In the sequel (especially to obtain estimates for solutions in H_s, with s non-integer) we will also use some elementary commutator type estimates as the following lemma concerning the operator Λ^s, $s \in \mathbb{R}^+$ (see e.g. [20, 21, 38]), where $\Lambda := (-\Delta)^{1/2}$.

Lemma 2.1. *For $s > 0$ and $1 < r \leq \infty$, and for smooth enough u and v*

$$\|\Lambda^s(uv)\|_{L^r} \leq C(\|u\|_{L^{p_1}}\|\Lambda^s v\|_{L^{q_1}} + \|v\|_{L^{p_2}}\|\Lambda^s u\|_{L^{q_2}}),$$

where $1/r = 1/p_1 + 1/q_1 = 1/p_2 + 1/q_2$ and C is a suitable positive constant.

2.2. **Existence results.** Concerning the Euler-Voigt equations we have the following result of global existence and uniqueness of solutions proved in [23, Thm. 3.1]: Therein the result is proved in two different ways, one with the contraction principle and the other one by means of the Galerkin method. The results explain the hyperbolic nature of the problem and are a starting point also to prove Gevrey regularity results. Here we give some remarks on one main technical point and also explain some (non strictly essentials) details on the external force.

Theorem 2.1. (Larios and Titi [23]) *Let $T > 0$ and let $u_0^\alpha \in H_m$ and $f \in C(-T, T; W^{m-1,6/5})$, with $\nabla \cdot f = 0$, for $m \geq 1$. Then, there exists a unique solution u^α of the Euler-Voigt equations* (2a)-(2b) *which belongs to $C^1[-T, T]; H_m)$. Moreover,*

$$\|u^\alpha(t)\|_{H^m} < C(\alpha, \|u_0\|_{H^m}, \sup_{-T<t<T} \|f(t)\|_{m-1,6/5}, T),$$

for all $t \in [-T, T]$.

Proof. We do not claim any originality in the result, but we just give an alternate proof of one main point, emphasizing also the role of the pressure. We use some classical techniques employed also in Beirão da Veiga [1] even if here due to the space-periodicity the proof is much simpler. We give the proof only for $m = 1$, because this is the most important step. The higher regularity can be obtained in the same way by a bootstrapping argument.

By using the Galerkin approach with approximate solutions u_m^α (where u_m^α is made with a finite Fourier expansion) it is elementary to show the *a-priori* estimate

$$u_m^\alpha \in L^\infty(-T, T; H_1).$$

By using standard tools it immediately follows that when $m \to +\infty$ the approximate functions u_m^α converge to u^α which is the unique solution solutions to (2a)-(2b) and which belongs to $L^\infty(-T, T; H_1)$. One difficulty consists in passing from boundedness to continuity with respect to the time variable. This can be obtained in an elementary manner as follows: First observe that the pressure p satisfies the Poisson equation

$$-\Delta p^\alpha = \nabla \cdot \left[(u^\alpha \cdot \nabla)\, u^\alpha\right].$$

Since $u^\alpha \in L^\infty(-T; T; H_1)$ we have that $\nabla \cdot [(u^\alpha \cdot \nabla)\, u^\alpha] \in L^\infty(-T, T; W^{-1,3/2})$, where $W^{-1,3/2} := (W^{1,3})'$ and consequently, by using the classical regularity theory for the Poisson equation in the periodic setting with zero mean value, we have

$$\nabla p^\alpha \in L^\infty(-T, T; L^{3/2}).$$

Then, by comparison it follows that

$$u_t^\alpha - \alpha^2 \Delta u_t^\alpha = -\nabla p^\alpha - (u^\alpha \cdot \nabla)\, u^\alpha + f \in L^\infty(-T, T; L^{6/5}),$$

and hence by using again the elliptic regularity to the equation $(\mathrm{I} - \alpha^2\Delta) u_t^\alpha = F$ and the Sobolev embedding we have

$$u_t^\alpha \in L^\infty(-T, T; W^{2,6/5}) \hookrightarrow L^\infty(-T, T; H^1).$$

By standard results it follows that u^α can be identified with a function continuous with values in H^1. Coming back (with the improved regularity on u^α) to the estimates on the convective term and on the gradient of the pressure they are both now in $C(-T, T; L^{6/5})$ and the same argument shows that

$$u_t^\alpha - \alpha^2 \Delta u_t^\alpha = -\nabla p^\alpha - (u^\alpha \cdot \nabla)\, u^\alpha + f \in C(-T, T; L^{6/5}),$$

hence that $u_t^\alpha \in C(-T, T; H^1)$. This ends the proof, since $\nabla \cdot u^\alpha = 0$. The high order regularity can be proved following the same approach. □

To conclude, we also recall the well-known results for the three-dimensional Euler equations

$$\partial_t u + (u \cdot \nabla)\, u + \nabla p = f, \tag{5a}$$

$$\nabla \cdot u = 0. \tag{5b}$$

i.e, when $\alpha = 0$. It is well known that if $u_0 \in H_s$, and $f \in L^1(0, \overline{T}, H_s)$ with $s > 5/2$, then there exists a unique solution to these equations in $C([0, T]; H_s) \cap C^1([0, T]; H_{s-1})$ for some finite time $0 < T \leq \overline{T}$ (see, e.g., the review in [31]). Let us recall that, contrary to the Euler-Voigt equations for which we know existence of global smooth unique solutions, it is an outstanding open problem determining whether smooth solutions exist globally in time or blow-up in finite time. In particular, the best know criterion for the 3D Euler equations is that of Beale-Kato-Majda. In [23, Thm. 5.2] it is obtained an interesting criterion linking the regularity of the Euler equations, with the dissipation of the Euler-Voigt equations as $\alpha \to 0$. More precisely it is shown that if

$$\sup_{t \in [0\overline{T}[} \limsup_{\alpha \to 0^+} \alpha^2 \|\nabla u^\alpha(t)\|^2 > 0,$$

then the Euler equations starting with the same initial datum u_0 of $\{u^\alpha\}_{\alpha>0}$ develop a singularity in $[0, \overline{T}]$. This result is a by product of a result of convergence as $\alpha \to 0^+$ of the Euler-Voigt equations. Consequently, the behavior as α vanishes is relevant also in view of applications of this model to computations for the study of the possible blow-up for the Euler system. In fact in [23, Thm. 5.2] it is also proved that the solutions u^α of the three-dimensional Euler-Voigt equations converge to the corresponding solutions u of the three-dimensional Euler equations, with respect to the norm $L^\infty(0, T; L^2(\mathbb{T}))$, and with initial data $u_0^\alpha = u_0 \in H^s(\mathbb{T})$, for $s > 5/2$. Our main interest is to study the sharp convergence, that in the same space of the initial datum. Here, the situation is a little bit different from the usual "Navier-Stokes $\to$ Euler limit", since the regularity requested on the initial conditions changes in the two systems: The Euler-Voigt requires one more derivative, in order to have uniform estimates in terms of α. As a by product of our results we also treat the behavior as the positive viscosity ν converges to zero and, when introducing the viscous problem, we also prove a result on time-periodic solutions.

2.3. **Time periodic solutions.** In this subsection we give some remarks on the proof of existence of time-periodic solutions. The results presented here are obtained with well-established techniques introduced in Prodi [36], taking the chance also of making some observations on the existence for the Navier-Stokes-Voigt equations. Since the results proved here hold true also in a bounded domain with Dirichlet boundary conditions, we use -just in this section- the notation

$$H_{1,\sigma} = \left\{ w : \Omega \to \mathbb{R}^3,\ w \in H^1(\Omega)^3, \quad \nabla \cdot w = 0,\ w_{|\partial\Omega} = 0 \right\},$$

and $H_{-1} := (H_{1,\sigma})'$. Since we are in a case very similar to the 2D Navier-Stokes equations (for the Naiver-Stokes-Voigt equations it is easy to prove existence and uniqueness of solutions for all times), we can work directly on the solutions, looking for a fixed point argument in the infinite dimensional space H_1. For the Navier-Stokes equations, the difference between 2D and 3D (regarding time-periodic solutions) are explained in [30, Ch. 4], where also the Galerkin method with the Brouwer fixed point is used to construct approximations to periodic solutions.

Proposition 2.1. *Let $\Omega \subset \mathbb{R}^3$ be a bounded, smooth, and open set; let $f \in L^2(0,T;H_{-1})$. Then, there exists at least a solution to the Navier-Stokes-Voigt equations*

$$\partial_t u - \alpha^2 \partial_t \Delta u + (u \cdot \nabla)\, u - \nu \Delta u + \nabla p = f \qquad \text{in } \Omega \times [0,T], \tag{6a}$$

$$\nabla \cdot u = 0 \qquad \text{in } \Omega \times [0,T], \tag{6b}$$

$$u = 0 \qquad \text{on } \partial\Omega \times]0,T], \tag{6c}$$

such that $u(0) = u(T) \in H_{1,\sigma}$.

Remark 2.1. *The same result holds also for $\Omega = \mathbb{T}$ with periodic boundary conditions. Moreover, the uniqueness of the time-periodic solutions still represent an open problem, posing the same difficulties as those well-known for the 2D Navier-Stokes equations.*

Proof. Since the proof is very standard, just a sketch of the proof of Proposition 2.1 is presented here, as a remark on what can be proved for the Navier-Stokes-Voigt equations. We observe that by the same usual methods based on Galerkin approximate functions and Aubin-Lions compactness tool we can construct a weak solution such that $u \in L^\infty(0,T;H_{1,\sigma})$ with

$$\|u(T)\|^2 + \alpha^2 \|\nabla u(T)\|^2 + \nu \int_0^T \|\nabla u(s)\|^2 ds \leq \|u_0\|^2 + \alpha^2 \|\nabla u_0\|^2 + C \int_0^T \|f(s)\|^2_{H_{-1}}\, ds.$$

Observe that the proof of this result can be obtained by making use of the comparison argument to prove that

$$u_t - \alpha^2 \Delta u_t \in L^2(0,T;H_{-1}),$$

hence that $u_t \in L^2(0,T;H_{1,\sigma})$ by the standard Lax-Milgram lemma set in the space $H_{1,\sigma}$. In particular, this proves that the solution $u \in C([0,T];H_{1,\sigma})$.

In order to prove existence of time-periodic solutions we have show that if $\|u(0)\|^2 + \alpha^2 \|\nabla u(0)\|^2 \leq R^2$ for a large enough $R > 0$, then the same bound holds true at $t = T$. Taking the inner product of (6a) with u, and by using the Poincarè inequality we get

$$\frac{d}{dt}\big(\|u\|^2 + \alpha^2 \|\nabla u\|^2\big) + c_1(\nu,\Omega,\alpha)\big(\|u\|^2 + \alpha^2 \|\nabla u\|^2\big) \leq c_2(\nu,\Omega) \|f\|^2_{H_{-1}}.$$

Consequently we have that

$$e^{c_1 T}(\|u(T)\|^2+\alpha^2\|\nabla u(T)\|^2) \leq \left(\|u(0)\|^2+\alpha^2\|\nabla u(0)\|^2\right) + \underbrace{c_2\int_0^T \|f(t)\|^2_{H_{-1}}\,dt}_{:=c_3}.$$

Therefore, to conclude it is sufficient to impose

$$R^2 \geq \frac{c_3}{1-e^{-c_1 T}},$$

to show that the solution satisfies

$$\|u(T)\|^2+\alpha^2\|\nabla u(T)\|^2 \leq R^2.$$

The proof follows by observing that the ball $B(0,R)\subset H_{1,\sigma}$ is a convex set in an Hilbert space, and it is therefore compact in the weak topology. Hence, by using Tychonov theorem we can argue that there exists a fixed point of the map $u_0 \to u(T)$, which is then a T-periodic solution to the Navier-Stokes-Voigt equations. $\square$

3. Convergence to the solutions of the Euler equations

In this section we prove the main result of the paper, that is a precise convergence result of smooth solutions of the Euler-Voigt equations to smooth solution of the Euler equations.

We start with the following result which is not optimal since one derivative is lost in the convergence. The technical reason, which can be easily understood, is that the H_m-estimates for the solution of the Euler-Voigt equation starting from a datum in H_m are not independent of α. In fact, in this case only the boundedness of $\alpha^2\|u^\alpha\|^2_{H_m}$ is known. To have estimates independent of α (and continuity up to $t=0$) one needs to assume more regularity on the initial datum. On the other hand, convergence in H_{m-2} is relatively easier to be obtained and the loss of two derivatives can be understood from the presence of the term Δu^α_t. We observe that this kind of results are obtained in [23]. New technical difficulties arise in our setting, since loosing one derivative is in some sense the best result when the initial datum is the same for both Euler and Euler-Voigt equations.

We first prove an intermediate result, since it represents the main technical point. Later on we will elaborate on the results which can be obtained when also the initial data can change (especially in terms of their regularity).

Theorem 3.1. *Let u be the solution to the Euler equations* (5a)-(5b) *with initial condition $u_0\in H_3$, and let u^α be the solution to* (2a)-(2b) *with initial condition $u^\alpha_0=u_0$. Let $T>0$ be a common time of existence for both u and u^α, with u, $u^\alpha\in C([0,T];H_3)\cap C^1([0,T];H_2)$. Then, for any sequence $\{\alpha_n\}$, with $\alpha_n>0$ and such that $\alpha_n\to 0$ as $n\to\infty$, it holds that*

$$\sup_{0<t<T}\|u^{\alpha_n}(t)-u(t)\|_{H_2}\to 0, \quad \text{as } n\to\infty.$$

Proof. The proof uses the smoothing tool introduced in [2, 3] and explained for the Euler equations in the periodic setting in [5], see also [32]. Let $u_0\in H_3$ and define $u_{0,\delta}$ as in (4), then $\nabla\cdot u_{0,\delta}=0$ and moreover, by direct computation,

$$\|u_{0,\delta}\|_{H_3}\leq C\|u_0\|_{H_3}, \qquad \|u_{0,\delta}\|_{H_4}\leq \frac{C}{\delta} \qquad \text{and} \quad \|u_{0,\delta}\|_{H_5}\leq \frac{C}{\delta^2}.$$

In addition, for s such that $0 < s < 3$, it also holds that $\|u_{0,\delta} - u_0\|_{H_s} \leq C\delta^{3-s}$. Let u_δ be the solution of the Euler equations with initial condition $u_{0,\delta}$, which we will call "regularized Euler equations." Then, in the interval $[0, T]$ the following relations hold true

$$\|u_\delta(t)\|_{H_3} < C \quad \text{and} \quad \|u_\delta(t)\|_{H_m} < \frac{C}{\delta^{m-3}}, \quad \text{with } m > 3. \tag{7}$$

We write

$$\|u^\alpha - u\|_{H_2} \leq \|u^\alpha - u_\delta\|_{H_2} + \|u_\delta - u\|_{H_2} =: I + II,$$

and we estimate $\|u^\alpha - u_\delta\|_{L^\infty(0,T;H_2)}$ and $\|u_\delta - u\|_{L^\infty(0,T;H_2)}$.

Estimate for $\|u^\alpha - u_\delta\|_{L^\infty(0,T;H_2)}$: We denote by $\omega_\delta^\alpha := u_\delta - u^\alpha$ the difference between the solution u^α of the Euler-Voigt equations (2a)-(2b) and u_δ. For simplicity we will use the notation $\omega := \omega_\delta^\alpha$. Thus, we get

$$\begin{aligned}\partial_t\omega - \alpha^2\partial_t\Delta\omega + \nabla(p_\delta - p^\alpha) &= -\alpha^2\partial_t\Delta u_\delta - (u_\delta\cdot\nabla)\,u_\delta + (u^\alpha\cdot\nabla)\,u^\alpha \\ &= -\alpha^2\partial_t\Delta u_\delta - (\omega\cdot\nabla)\,u_\delta - (u^\alpha\cdot\nabla)\,\omega,\end{aligned}$$

and by taking the H_2-inner product with ω we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}(\|\omega\|_{H_2}^2 + \alpha^2\|\nabla\omega\|_{H_2}^2) \leq &\alpha^2|(\Delta\partial_t u_\delta, \omega)_{H_2}| + |((\omega\cdot\nabla)\,u_\delta, \omega)_{H_2}| \\ &+ |((u^\alpha\cdot\nabla)\,\omega, \omega)_{H_2}|.\end{aligned} \tag{8}$$

Next, we estimate the first term from the right-hand side. Let us recall that u_δ is a solution to the (regularized) Euler equations, so it follows that

$$\begin{aligned}\alpha^2\big|(\Delta\partial_t u_\delta, \omega)_{H_2}\big| &= \alpha^2\big|(\partial_t u_\delta, \Delta\omega)_{H_2}\big| \\ &\leq \alpha^2\big|((u_\delta\cdot\nabla)\,u_\delta, \Delta\omega)_{H_2}\big| + \alpha^2\big|(\nabla p, \Delta\omega)_{H_2}\big| \\ &= \alpha^2\big|(\nabla\big[(u_\delta\cdot\nabla)\big]\,u_\delta, \nabla\omega)_{H_2}\big|,\end{aligned}$$

due to periodicity and also to the incompressibility constraint. By using the regularity of the solution of the Euler equations, we have then

$$\begin{aligned}\alpha^2|(\nabla\big[(u_\delta\cdot\nabla)\,u_\delta\big], \nabla\omega)_{H_2}| &\leq C\alpha^2(\|u_\delta\|_{H_3}^2 + \|u_\delta\|_{H_2}\|u_\delta\|_{H_4})\|\nabla\omega\|_{H_2} \\ &\leq C\alpha^2(\|u_\delta\|_{H_3}^2 + \frac{\|u_\delta\|_{H_2}}{\delta})\|\nabla\omega\|_{H_2} \\ &\leq C\frac{\alpha^2}{\delta}\|\nabla\omega\|_{H_2},\end{aligned}$$

where we are supposing for simplicity that $0 < \delta < 1$ (since we will main use values of δ close to zero). By using classical estimates on the convective term as in Kato [19, Eq. (2.1)-(2.2')], we estimate the other terms from the right-hand side of (8) as follows:

$$|((u^\alpha\cdot\nabla)\,\omega), \omega)_{H_2}| \leq \|u^\alpha\|_{H_3}\|\omega\|_{H_2}^2 \quad \text{and} \quad |((\omega\cdot\nabla)\,u_\delta), \omega)_{H_2}| \leq \|u_\delta\|_{H_3}\|\omega\|_{H_2}^2.$$

Collecting the above estimates, using the bounds for the solution of the (regularized) Euler equations, and with Schwarz inequality we get

$$\frac{d}{dt}(\|\omega\|_{H_2}^2 + \alpha^2\|\nabla\omega\|_{H_2}^2) \leq C(\|\omega\|_{H_2}^2 + \alpha^2\|\nabla\omega\|_{H_2}^2) + C\frac{\alpha^2}{\delta^2}.$$

Thus, by using the Gronwall's lemma we infer that

$$
\begin{aligned}
\|u^\alpha - u_\delta\|^2_{L^\infty(0,T;H_2)} &\leq \big(\frac{\alpha^2}{\delta^2}T + \|u_0 - u_{0,\delta}\|^2_{H_2} + \alpha^2\|\nabla(u_0 - u_{0,\delta})\|^2_{H_2}\big)Ce^{CT} \\
&\leq \big(\frac{\alpha^2}{\delta^2}T + \delta^2 + \alpha^2\|u_{0,\delta} - u_0\|^2_{H_3}\big)C_1(T).
\end{aligned}
\tag{9}
$$

Estimate for $\|u_\delta - u\|_{L^\infty(0,T;H_2)}$: Here, we take the H_2-energy estimate for $\omega_\delta := u_\delta - u$ and we find (since they are both solutions to the Euler equations with different initial data)

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|\omega_\delta\|^2_{H_2} &\leq |((\omega_\delta\cdot\nabla)\,u_\delta, \omega_\delta)_{H_2}| + |((u\cdot\nabla)\,\omega_\delta, \omega_\delta)_{H_2}| \\
&\leq C(\|u\|_{H_3} + \|u_\delta\|_{H_3})\|\omega_\delta\|^2_{H_2}.
\end{aligned}
$$

that is $\|\omega_\delta\|_{H_2} \leq \|\omega_{0,\delta}\|_{H_2}e^{CT} \leq \delta\, C_2(T)$, and finally

$$
\|u_\delta - u\|_{L^\infty(0,T;H_2)} \leq \delta\, C_2(T). \tag{10}
$$

As a consequence of the estimates (9)-(10) we can conclude that

$$
\|u^\alpha - u\|_{L^\infty(0,T;H_2)} \leq C\big(\delta^2 + \frac{\alpha^2}{\delta^2}T + \alpha^2\|\nabla(u_{0,\delta} - u_0)\|^2_{H_2}\big)^{1/2}C(T),
$$

where $C(T) = \max\big\{C_1^{1/2}(T), C_2(T)\big\}$. Now, by taking $\delta = \delta_n$ such that both δ_n and $\frac{\alpha_n}{\delta_n}$ go to zero as n goes to infinity, we obtain the required convergence in the H_2-norm. In particular, it follows that $\|u^\alpha - u\|_{L^\infty(0,T;H_2)} = O(\sqrt{\alpha})$. □

By using exactly the same arguments one can easily prove, more generally, the following result in smoother spaces.

Theorem 3.2. *Let* u *be the solution to the Euler equations* (5a)-(5b) *with initial condition* $u_0 \in H_{m+1}$*, with* m *integer such that* $m \geq 2$*, and let* u^α *be a solution to the Euler-Voigt equations* (2a)-(2b) *with initial condition* $u_0^\alpha = u_0$*. Let* $T > 0$ *be a common time of existence for* u *and* u^α*, with* $u^\alpha, u \in C([0,T];H_{m+1}) \cap C^1([0,T];H_m)$*. Then, for any sequence* $\{\alpha_n\}$ *with* $\alpha_n > 0$ *and such that* $\alpha_n \to 0$ *as* $n \to \infty$*, it holds that*

$$
\sup_{0<t<T}\|u^{\alpha_n}(t) - u(t)\|_{H_m} \to 0, \qquad \textit{as } n \to \infty.
$$

We arrive now to the main result of the paper which shows the precise convergence in terms of the initial data and of the space without unnatural loss of regularity.

Theorem 3.3. *Consider the Euler equations* (5a)-(5b) *with initial condition* $u_0 \in H_3$*, and let* $T > 0$ *be a finite time of existence for the solution* $u \in C([0,T];H_3) \cap C^1([0,T];H_2)$*. Let* $u^{\alpha,\beta}$ *be a solution to the Euler-Voigt equations* (2a)-(2b)*, with initial condition* u_0^β*, such that*

$$
\begin{aligned}
&i) \quad u_0^\beta \in H_4, \quad \textit{for } \beta > 0 \\
&ii) \quad \|u_0^\beta - u_0\|_{H_3} \to 0 \quad \textit{as} \quad \beta \to 0.
\end{aligned}
\tag{11}
$$

Then, for any sequence $\{\beta_n\}$ with $\beta_n>0$ and such that $\beta_n\to 0$ as $n\to\infty$ there exists $\{\alpha_n\}$ with $\alpha_n>0$ and converging to zero such that

$$\sup_{0<t<T}\|u^{\alpha_n,\beta_n}(t)-u(t)\|_{H_3}\to 0,\qquad \text{as } n\to\infty.$$

To prove the Theorem 3.3 we need the following preliminary lemma showing that we can construct the solution u^{α_n,β_n} in a time interval independent of $n\in\mathbb{N}$, and this would be enough in order to get weak convergence results by using the classical compactness methods, even if we are interested in strong convergence.

Lemma 3.1. *Under the hypotheses of Theorem 3.3, it follows that, for any positive sequence $\{\beta_n\}$ such that $\beta_n\to 0$, as $n\to\infty$, we can find a positive sequence $\{\alpha_n\}$ with $\alpha_n\to 0$ such that $\|u^{\alpha_n,\beta_n}\|_{L^\infty(0,T;H_3)}$ results bounded uniformly in $n\in\mathbb{N}$.*

Proof. Consider the Euler-Voigt equations (2a)-(2b) with initial data u_0^β. Taking the H_3-inner product with $u^{\alpha,\beta}$, and with the usual inequalities for the convective term we obtain

$$\frac{d}{dt}(\|u^{\alpha,\beta}\|_{H_3}^2+\alpha^2\|\nabla u^{\alpha,\beta}\|_{H_3}^2)\le C\big(\|u^{\alpha,\beta}\|_{H_3}^2+\alpha^2\|\nabla u^{\alpha,\beta}\|_{H_3}^2\big)^{\frac{3}{2}}.$$

Thus, we deduce that

$$\|u^{\alpha,\beta}(t)\|_{H_3}^2+\alpha^2\|\nabla u^{\alpha,\beta}(t)\|_{H_3}^2\le\frac{\|u_0^\beta\|_{H_3}^2+\alpha^2\|\nabla u_0^\beta\|_{H_3}^2}{\Big[1-Ct\big(\|u_0^\beta\|_{H_3}^2+\alpha^2\|\nabla u_0^\beta\|_{H_3}^2\big)^{1/2}\Big]^2},$$

with $t\in[0,T]$. Thanks to the properties $i)$ and $ii)$ in (11), it follows that, letting $\beta\to 0$, then $\|u_0^\beta\|_{H_3}\to\|u_0\|_{H_3}$. Next, we can choose $\alpha\to 0$ such that $\alpha\|\nabla u_0^\beta\|_{H_3}$ remains bounded (for instance choose $\alpha=O(\|u_0^\beta\|_{H_4}^{-1})$.) in this way $\|u^{\alpha,\beta}\|_{L^\infty(0,T;H_3)}$ results to be uniformly bounded as well, in a time interval $[0,T]$ independent of β. □

We can now give the proof of Theorem 3.3.

Proof of Theorem 3.3. To estimate $\|u^{\alpha,\beta}-u\|_{L^\infty(0,T;H_3)}$, we write

$$\begin{aligned}\|u^{\alpha,\beta}-u\|_{H_3}&\le\|u^{\alpha,\beta}-u^\beta\|_{H_3}+\|u^\beta-u\|_{H_3}=:I+II,\\&\le\|u^{\alpha,\beta}-u_\delta\|_{H_3}+\|u_\delta-u^\beta\|_{H_3}+\|u^\beta-u\|_{H_3}=:I_1+I_2+II,\end{aligned}$$

where u^β is the solution of the Euler equations with initial data u_0^β and u_δ is the solution of the Euler equations starting from the regularized initial datum $u_{0,\delta}$. (Note that, Lemma 3.1 applies also to u^β, and consequently $\|u^\beta\|_{H_3}$ results to be uniformly bounded with respect to β).

Estimate for $\|u^{\alpha,\beta}-u_\delta\|_{L^\infty(0,T;H_3)}$: By setting $\omega:=u_\delta-u^{\alpha,\beta}$, we get

$$\partial_t\omega+\alpha^2\partial_t\Delta\omega+\nabla(p_\delta-p^{\alpha,\beta})=-\alpha^2\partial_t\Delta u_\delta-(\omega\cdot\nabla)\,u_\delta-(u^{\alpha,\beta}\cdot\nabla)\,\omega.$$

Taking the H_3-inner product with ω, and recalling that u^δ is solution to the (regularized) Euler equations, we arrive at

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}(\|\omega\|_{H_3}^2+\alpha^2\|\nabla\omega\|_{H_3}^2)\le{}&|((\omega\cdot\nabla)\,u_\delta,\omega)_{H_3}|+|((u^{\alpha,\beta}\cdot\nabla)\,\omega),\omega)_{H_3}|\\&+\alpha^2|(\nabla(u_\delta\cdot\nabla)\,u_\delta,\nabla\omega)_{H_3}|.\end{aligned}$$

From [19, Eq. (2.2)], we get

$$|((u^{\alpha,\beta}\cdot\nabla)\,\omega,\omega)_{H_3}| \le \|u^{\alpha,\beta}\|_{H_3}\|\omega\|^2_{H_3}.$$

With direct computations we obtain

$$\begin{aligned}\alpha^2|(\nabla(u_\delta\cdot\nabla)\,u_\delta,\nabla\omega)_{H_3}| &\le C\alpha^2(\|u_\delta\|_{H_3}\|u_\delta\|_{H_4}+\|u_\delta\|_{H_2}\|u_\delta\|_{H_5})\|\nabla\omega\|_{H_3}\\ &\le C\alpha^2(\delta^{-1}+\delta^{-2})\|\nabla\omega\|_{H_3}=\alpha^2\check{C}(\delta)\|\nabla\omega\|_{H_3},\end{aligned}$$

and also

$$|((\omega\cdot\nabla)u_\delta,\omega)_{H_3}| \le \|u_\delta\|_{H_3}\|\omega\|^2_{H_3}+\|u_\delta\|_{H_4}\|\omega\|_{L^\infty}\|\omega\|_{H_3}.$$

Now, to estimate the second term on the right-hand in $H_{s'}$, with $\frac{3}{2}<s'<2$, we employ usual techniques and Lemma 2.1 to get

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}(\|\omega\|^2_{H_{s'}}+\alpha^2\|\nabla\omega\|^2_{H_{s'}}) \le &(\|u_\delta\|_{H_{s'+1}}+\|u^{\alpha,\beta}\|_{H_{s'+1}})\|\omega\|^2_{H_{s'}}\\ &+\alpha^2\big|(\nabla\big[(u_\delta\cdot\nabla)\,u_\delta\big],\nabla\omega)_{H_{s'}}\big|,\end{aligned}\tag{12}$$

and again

$$\begin{aligned}\alpha^2\big|(\nabla\big[(u_\delta\cdot\nabla)\,u_\delta\big],\nabla\omega)_{H_{s'}}\big| &\le C\alpha^2(\|u_\delta\|_{H_{s'+1}}\|u_\delta\|_{H_3}+\|u_\delta\|_{H_{s'+2}}\|u_\delta\|_{H_2})\|\nabla\omega\|_{H_{s'}}\\ &\le C\alpha^2\big(\|u_\delta\|_{H_{s'+1}}+\delta^{1-s'}\big)\|\nabla\omega\|_{H_{s'}}\\ &\le C\alpha^2\big(1+\delta^{1-s'}\big)=\alpha^2\widetilde{C}(\delta)\|\nabla\omega\|_{H_{s'}}.\end{aligned}$$

Then, it follows

$$\frac{d}{dt}(\|\omega\|^2_{H_{s'}}+\alpha^2\|\nabla\omega\|^2_{H_{s'}}) \le C(\|\omega\|^2_{H_{s'}}+\alpha^2\|\nabla\omega\|^2_{H_{s'}})+\alpha^2\widetilde{C}^2(\delta).$$

Thus, from the above differential inequality we get an estimate for $\|\omega\|^2_{H_{s'}}$, and taking the L^∞-norm on $[0,T]$ we find

$$\|\omega\|^2_{L^\infty(0,T;H_{s'})} \le \big(\|\omega_0\|^2_{H_{s'}}+\alpha^2\|\nabla\omega_0\|^2_{H_{s'}}+\alpha^2\widetilde{C}^2(\delta)T\big)C(T).$$

Summarizing the previous estimates, we have

$$\begin{aligned}\frac{d}{dt}(\|\omega\|^2_{H_3}+\alpha^2\|\nabla\omega\|^2_{H_3}) \le\ & C(\|u_\delta\|_{H_3}+\|u^{\alpha,\beta}\|_{H_3})\|\omega\|^2_{H_3}+\alpha^2\check{C}(\delta)\|\nabla\omega\|_{H_3}\\ &+\frac{C(T)^{\frac{1}{2}}}{\delta}\big(\|\omega_0\|^2_{H_{s'}}+\alpha^2(\|\omega_0\|^2_{H_{s'+1}}+\widetilde{C}^2(\delta)\,T)\big)^{\frac{1}{2}}\|\omega\|_{H_3}.\end{aligned}$$

To handle the term $\|\omega_0\|_{H_{s'}}$ from the right-hand side we write

$$\|\omega_0\|_{H_{s'}}=\|u_{0,\delta}-u_0^\beta\|_{H_{s'}} \le \|u_{0,\delta}-u^\beta_{0,\delta}\|_{H_{s'}}+\|u^\beta_{0,\delta}-u^\beta_0\|_{H_{s'}}.\tag{13}$$

where $u^\beta_{0,\delta}$ is the regularization of the initial datum u^β_0. Expanding $\|u^\beta_{0,\delta}-u_{0,\delta}\|^2_{H_{s'}}$ and $\|u^\beta_{0,\delta}-u^\beta_0\|_{H_{s'}}$ in terms of their Fourier coefficients, we have that

a) $\|u^\beta_{0,\delta}-u_{0,\delta}\|^2_{H_{s'}} \le C\sum_{1\le|k|\le\frac{1}{\delta}}|k|^{2s'}|u^\beta_{0,k}-u_{0,k}|^2 \le C\delta^{2(3-s')}\|u^\beta_0-u_0\|^2_{H_3}$,

b) $\|u^\beta_{0,\delta}-u^\beta_0\|^2_{H_{s'}} \le C\sum_{|k|>\frac{1}{\delta}}|k|^{2s'}|u^\beta_{0,k}|^2 \le C\delta^{2(3-s')}\|u^\beta_0\|^2_{H_3}$.

Therefore, using (13) and the above inequalities, we obtain

$$\|\omega_0\|^2_{H_{s'}} \le C\delta^{6-2s'}(\|u^\beta_0-u_0\|^2_{H_3}+\|u^\beta_0\|^2_{H_3}).\tag{14}$$

Analogously, we get $\|\omega_0\|^2_{H_{s'+1}} \le C\delta^{4-2s'}(\|u_0^\beta - u_0\|^2_{H_3} + \|u_0^\beta\|^2_{H_3})$. Then, the differential inequality for the H_3-norm becomes

$$\begin{aligned}\frac{d}{dt}(\|\omega\|^2_{H_3} + \alpha^2\|\nabla\omega\|^2_{H_3}) \le & C(\|u_\delta\|_{H_3} + \|u^{\alpha,\beta}\|_{H_3})\|\omega\|^2_{H_3} \\ & + \alpha^2\check{C}(\delta)\|\nabla\omega\|_{H_3} + \widehat{C}(T,\alpha,\beta,\delta)\|\omega\|_{H_3},\end{aligned}$$

where

$$\widehat{C}(T,\alpha,\beta,\delta) := \Big(\delta^{2-2s'}(\delta^2+\alpha^2)\big(\|u_0^\beta - u_0\|^2_{H_3} + \|u_0^\beta\|^2_{H_3}\big) + \frac{\alpha^2}{\delta^2}\widetilde{C}^2(\delta)T\Big)^{\frac12} C^{\frac12}(T).$$

After some manipulations we get

$$\begin{aligned}\frac{d}{dt}(\|\omega\|^2_{H_3} + \alpha^2\|\nabla\omega\|^2_{H_3}) \le & C(\|u_\delta\|_{H_3} + \|u^{\alpha,\beta}\|_{H_3} + 1)(\|\omega\|^2_{H_3} + \alpha^2\|\nabla\omega\|^2_{H_3}) \\ & + \alpha^2\check{C}^2(\delta) + \widehat{C}^2(T,\alpha,\beta,\delta),\end{aligned}$$

and by the Gronwall's inequality it follows that

$$\begin{aligned}\|u_\delta - u^{\alpha,\beta}\|^2_{L^\infty(0,T;H_3)} \le & \Big((\alpha^2\check{C}^2(\delta) + \widehat{C}^2(T,\alpha,\beta,\delta))T \\ & + \|\omega_0\|^2_{H_3} + \alpha^2\|\nabla\omega_0\|^2_{H_3}\Big)e^{C(T,\alpha,\beta,\delta)},\end{aligned} \tag{15}$$

where $C(T,\alpha,\beta,\delta) := \int_0^T C(\|u_\delta(s)\|_{H_3} + \|u^{\alpha,\beta}(s)\|_{H_3} + 1)\,ds$. Now, we have that

$$\|\nabla\omega_0\|^2_{H_3} \le C\|\nabla(u^\beta_{0,\delta} - u_{0,\delta})\|^2_{H_3} + C\|\nabla(u^\beta_{0,\delta} - u^\beta_0)\|^2_{H_3}.$$

Then, expanding $\|\nabla(u^\beta_{0,\delta} - u_{0,\delta})\|^2_{H_3}$ in terms of its Fourier coefficients, we find

$$\begin{aligned}\|\nabla(u^\beta_{0,\delta} - u_{0,\delta})\|^2_{H_3} \le & C\sum_{1\le|k|\le\frac{1}{\delta}} |k|^8|u^\beta_{0,k} - u_{0,k}|^2 \\ \le & \frac{C}{\delta^2}\sum_{1\le|k|\le\frac{1}{\delta}} |k|^6|u^\beta_{0,k} - u_{0,k}|^2 \le \frac{C}{\delta^2}\|u_0^\beta - u_0\|^2_{H_3},\end{aligned}$$

and it follows that

$$\begin{aligned}\alpha^2\|\nabla\omega_0\|^2_{H_3} \le & C\frac{\alpha^2}{\delta^2}\|u_0^\beta - u_0\|^2_{H_3} + C\alpha^2\|u^\beta_{0,\delta} - u^\beta_0\|^2_{H_4} \\ \le & C\frac{\alpha^2}{\delta^2}\|u_0^\beta - u_0\|^2_{H_3} + C\alpha^2\|u^\beta_0\|^2_{H_4}.\end{aligned} \tag{16}$$

Hence, by using $\|\omega_0\|^2_{H_3} \le C\|u_{0,\delta} - u_0\|^2_{H_3} + C\|u_0^\beta - u_0\|^2_{H_3}$ and (16), the estimate (15) becomes

$$\begin{aligned}& \|u_\delta - u^{\alpha,\beta}\|^2_{L^\infty(0,T;H_3)} \\ & \quad \le \Big(\big(\alpha^2\check{C}^2(\delta) + \widehat{C}^2(T,\alpha,\beta,\delta)\big)T + C\|u_{0,\delta} - u_0\|^2_{H_3} \\ & \qquad + C(1 + \frac{\alpha^2}{\delta^2})\|u_0^\beta - u_0\|^2_{H_3} + C\alpha^2\|u_0^\beta\|^2_{H_4}\Big)e^{C(T,\alpha,\beta,\delta)}.\end{aligned} \tag{17}$$

Estimate for $\|u_\delta - u^\beta\|_{L^\infty(0,T;H_3)}$: Writing the H_3-energy estimate for $\omega^\beta_\delta := u_\delta - u^\beta$ and using, for the sake of brevity, ω instead of ω^β_δ we have that

$$\frac12\frac{d}{dt}\|\omega\|^2_{H_3} \le |((\omega\cdot\nabla)\,u_\delta, \omega)_{H_3}| + |((u^\beta\cdot\nabla)\,\omega,\, \omega)_{H_3}|,$$

and by the usual estimates

$$\frac{d}{dt}\|\omega\|_{H_3}^2 \leq C(\|u_\delta\|_{H_3} + \|u^\beta\|_{H_3})\|\omega\|_{H_3}^2 + C\|u_\delta\|_{H_4}\|\omega\|_{L^\infty}\|\omega\|_{H_3}. \tag{18}$$

Next, to estimate the second term on the right-hand side of (18) we use again an $H_{s'}$-energy inequality, with $\frac{3}{2} < s' < 2$. Thus, arguing as in the derivation of (12) we get

$$\frac{d}{dt}\|\omega\|_{H_{s'}}^2 \leq C(\|u_\delta\|_{H_{s'+1}} + \|u^\beta\|_{H_{s'+1}})\|\omega\|_{H_{s'}}^2,$$

and using relation (14), we infer that

$$\|\omega\|_{L^\infty(0,T;H_{s'})} \leq \|\omega_0\|_{H_{s'}}C(T) \leq \delta^{3-s'}(\|u_0^\beta\|_{H_3}^2 + \|u_0^\beta - u_0\|_{H_3}^2)^{1/2}C(T).$$

Thus, relation (18) becomes

$$\frac{d}{dt}\|\omega\|_{H_3} \leq C(\|u^\delta\|_{H_3} + \|u^\beta\|_{H_3} + 1)\|\omega\|_{H_3} + \overset{\curlywedge}{C}(T,\beta)\delta^{2-s'},$$

where $\overset{\curlywedge}{C}(T,\beta) := C(T)(\|u_0^\beta\|_{H_3} + \|u_0^\beta - u_0\|_{H_3})^{1/2}$. Therefore, applying the Gronwall lemma, we deduce that

$$\begin{aligned}\|u_\delta - u^\beta\|_{L^\infty(0,T;H_3)} \leq &\big(\|u_{0,\delta} - u_0\|_{H_3} + \|u_0^\beta - u_0\|_{H_3} \\ &+ \delta^{2-s'}\overset{\curlywedge}{C}(T,\beta)T\big)e^{C(T,\beta,\delta)},\end{aligned} \tag{19}$$

with $C(T,\beta,\delta) := \int_0^T C(\|u_\delta(s)\|_{H_3} + \|u^\beta(s)\|_{H_3} + 1)\, ds$.

Estimate for $\|u^{\alpha,\beta} - u^\beta\|_{L^\infty(0,T;H_3)}$:

Using (17) and (19) we have that

$$\begin{aligned}\|u^{\alpha,\beta} - u^\beta\|_{L^\infty(0,T;H_3)}^2 &\leq C\big(\|u^{\alpha,\beta} - u_\delta\|_{L^\infty(0,T;H_3)}^2 + \|u_\delta - u^\beta\|_{L^\infty(0,T;H_3)}^2\big) \\ &\leq C\Big[\big(\alpha^2\check{C}^2(\delta) + \widehat{C}^2(T,\alpha,\beta,\delta)\big)T + C\|u_{0,\delta} - u_0\|_{H_3}^2 \\ &\qquad + C(1 + \frac{\alpha^2}{\delta^2})\|u_0^\beta - u_0\|_{H_3}^2 + \alpha^2\|u_0^\beta\|_{H_4}^2 \\ &\qquad + \delta^{4-2s'}\overset{\curlywedge}{C}^2(T,\beta)T^2\Big]e^{2C(T,\beta,\delta)+C(T,\alpha,\beta,\delta)}.\end{aligned} \tag{20}$$

Estimate for $\|u^\beta - u\|_{L^\infty(0,T;H_3)}$: We split II as follows

$$II \leq \|u^\beta - u_\delta\|_{H_3} + \|u_\delta - u\|_{H_3}. \tag{21}$$

Consider the first term on the right hand-side of (21). It follows that the difference $\|u_\delta - u^\beta\|_{L^\infty(0,T;H_3)}$ can be estimated as done in (19). For the second term on the right hand-side of (21), we can use the same [32, Eq. (23)]. Hence, we actually get

$$\|u_\delta - u\|_{L^\infty(0,T;H_3)} \leq (\|u_{0,\delta} - u_0\|_{H_3} + \delta^{2-s'}T)C(T).$$

with $\frac{3}{2} < s' < 2$. Now, multiplying the right-hand side of the latter inequality by $e^{C(T,\beta,\delta)}$, and then adding to (19), we get the estimate

$$\begin{aligned}\|u^\beta - u\|_{L^\infty(0,T;H_3)} \leq &C\Big(\|u_{0,\delta} - u_0\|_{H_3}\big(C(T) + C\big) + \|u_0^\beta - u_0\|_{H_3} \\ &+ \delta^{2-s'}\big(C(T) + \overset{\curlywedge}{C}(T,\beta)\big)T\Big)e^{C(T,\beta,\delta)}.\end{aligned} \tag{22}$$

Note that, the terms $C(T), \overset{\curlywedge}{C}(T,\beta)$, and $C(T,\beta,\delta)$ are bounded in terms of δ and β. Then, for any positive sequence $\{\beta_n\}$, such that $\beta_n \to 0$ as $n \to \infty$, letting $\delta \to 0$, we obtain

$$\|u^{\beta_n} - u\|_{L^\infty(0,T;H_3)} \to 0 \qquad \text{as } n \to \infty.$$

Furthermore, by relation (20), letting $\alpha_n \to 0$, and taking a sequence $\delta_n \to 0$, such that $\|u^{\beta_n}_{0,\delta_n} - u_0\|_{H_3}$, $\alpha_n\|u^{\beta_n}_0\|_{H_4}$, $\frac{\alpha_n}{\delta_n^2}$, $\frac{\alpha_n}{\delta_n}$, and $\frac{\alpha_n}{\delta_n^{s'-1}}$ go to zero as n goes to infinity, we find

$$\|u^{\alpha_n,\beta_n} - u^{\beta_n}\|_{L^\infty(0,T;H_3)} \to 0 \quad \text{as } n \to \infty.$$

Hence, we have that

$$\|u^{\alpha_n,\beta_n} - u\|_{L^\infty(0,T;H_3)} \to 0 \quad \text{as } n \to \infty,$$

and the thesis follows. $\square$

4. Convergence of the solutions of the Navier-Stokes-Voigt equations

Combining the results of the previous section with similar computations, we study also the behavior of solutions in terms of the viscosity. We are still set in the space-periodic case and, for simplicity we assume $f = 0$. Next, we state a convergence result for solutions of the Navier-Stokes-Voigt equations to the corresponding solutions of the Euler equations.

Theorem 4.1. *Consider the Euler equations* (5a)-(5b) *with initial condition* $u_0 \in H_3$*, and let* $T > 0$ *be a finite time of existence for the solution* $u \in C([0,T];H_3)$*. Let* $u^{\alpha,\beta,\nu}$ *be a solution to the Navier-Stokes-Voigt equations* (6a)-(6c)*, with initial datum* u_0^β*, satisfying the properties* i) *and* ii) *in* (11)*. Then, for any choice of positive sequences* $\{\beta_n\}$ *and* $\{\nu_n\}$*, both converging to zero as* $n \to \infty$*, there exists a positive sequence* $\{\alpha_n\}$ *converging to zero as* $n \to \infty$*, such that*

$$\sup_{0<t<T} \|u^{\alpha_n,\beta_n,\nu_n} - u\|_{H_3} \to 0, \quad \text{as } n \to \infty.$$

Proof. In the following, $u^{\alpha,\beta}$ and $u^{\alpha,\beta,\nu}$ will indicate the solutions of the Euler-Voigt and the Navier-Stokes-Voigt equations (with the same initial datum u_0^β) respectively. To estimate $\|u^{\alpha,\beta,\nu} - u\|_{L^\infty(0,T;H_3)}$, we take into account the following terms

$$\begin{aligned}\|u^{\alpha,\beta,\nu} - u\|_{L^\infty(0,T;H_3)} \leq & \|u^{\alpha,\beta,\nu} - u^\alpha_\delta\|_{L^\infty(0,T;H_3)} \\ & + \|u^\alpha_\delta - u\|_{L^\infty(0,T;H_3)} =: I + II,\end{aligned} \tag{23}$$

where u^α_δ is the solution of the Euler-Voigt equations with respect to the regularized initial datum $u_{0,\delta}$. Then, the term II will be split as follows

$$II \leq \|u^\alpha_\delta - u^{\alpha,\beta}\|_{L^\infty(0,T;H_3)} + \|u^{\alpha,\beta} - u\|_{L^\infty(0,T;H_3)} =: II_1 + II_2.$$

The above splitting is probably not the simplest one, but it is the most convenient to employ the results proved in the previous section.

Estimate for $\|u^{\alpha,\beta,\nu} - u^\alpha_\delta\|_{L^\infty(0,T;H_3)}$: Setting $\omega^{\alpha,\beta,\nu}_\delta := u^\alpha_\delta - u^{\alpha,\beta,\nu}$ (as usual for the difference $\omega^{\alpha,\beta,\nu}_\delta$, we drop the symbols α, β, δ, and ν), we get

$$\partial_t\omega - \alpha^2\partial_t\Delta\omega - \nu\Delta\omega + \nabla(p^\alpha_\delta - p^{\alpha,\nu}) = -\nu\Delta u^\alpha_\delta - (\omega\cdot\nabla)\,u^\alpha_\delta - (u^{\alpha,\beta,\nu}\cdot\nabla)\,\omega.$$

Taking the H_3-inner product of the above relation with ω, with the same inequalities employed in the previous sections we arrive at

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}(\|\omega\|_{H_3}^2+\alpha^2\|\nabla\omega\|_{H_3}^2)+\frac{\nu}{2}\|\nabla\omega\|_{H_3}^2 \leq \frac{\nu}{2}\|\nabla u_\delta^\alpha\|_{H_3}^2 \\ +\big((\|u_\delta^\alpha\|_{H_3}+\|u^{\alpha,\beta,\nu}\|_{H_3})\|\omega\|_{H_3}+\|u_\delta^\alpha\|_{H_4}\|\omega\|_{L^\infty}\big)\|\omega\|_{H_3}.\end{aligned} \tag{24}$$

In order to estimate $\|\omega\|_{L^\infty(\mathbb{T}^3)}$, we use again the same tool (with the $H_{s'}$-energy inequality, for $\frac{3}{2}<s'<2$.) By Lemma 2.1, we get

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}(\|\omega\|_{H_{s'}}^2+\alpha^2\|\nabla\omega\|_{H_{s'}}^2) &\leq C(\|u_\delta^\alpha\|_{H_{s'+1}}+\|u^{\alpha,\beta,\nu}\|_{H_{s'+1}})\|\omega\|_{H_{s'}}^2+\frac{\nu}{2}\|\nabla u_\delta^\alpha\|_{H_{s'}}^2 \\ &\leq C(\|\omega\|_{H_{s'}}^2+\alpha^2\|\nabla\omega\|_{H_{s'}}^2)+C\nu.\end{aligned}$$

Then, it follows that

$$\|\omega\|_{L^\infty(0,T;H_{s'})}^2 \leq \big((\|\omega_0\|_{H_{s'}}^2+\alpha^2\|\nabla\omega_0\|_{H_{s'}}^2)+C\nu T\big)C(T).$$

Now, it holds that for $0\leq k\leq 3$

$$\begin{aligned}\|\omega_0\|_{H_k}^2=\|u_{0,\delta}-u_0^\beta\|_{H_k}^2 &\leq C\|u_{0,\delta}-u_{0,\delta}^\beta\|_{H_k}^2+C\|u_{0,\delta}^\beta-u_0^\beta\|_{H_k}^2 \\ &\leq C\delta^{6-2k}(\|u_0^\beta-u_0\|_{H_3}^2+\|u_0^\beta\|_{H_3}^2).\end{aligned} \tag{25}$$

In particular, for $k=s'$ and $k=s'+1$ we get

$$\begin{aligned}&a)\ \|u_{0,\delta}-u_0^\beta\|_{H_{s'}}^2 \leq C\delta^{6-2s'}(\|u_0^\beta-u_0\|_{H_3}^2+\|u_0^\beta\|_{H_3}^2),\\ &b)\ \|u_{0,\delta}-u_0^\beta\|_{H_{s'+1}}^2 \leq C\delta^{4-2s'}(\|u_0^\beta-u_0\|_{H_3}^2+\|u_0^\beta\|_{H_3}^2).\end{aligned}$$

Consequently

$$\|\omega\|_{L^\infty(0,T;H_{s'})}^2 \leq \big(\delta^{4-2s'}(\delta^2+C\alpha^2)(\|u_0^\beta-u_0\|_{H_3}^2+\|u_0^\beta\|_{H_3}^2)+C\nu T\big)C(T).$$

We use the above inequality and the bound $\|\nabla u_\delta^\alpha\|_{H_3}\leq C\|u^\alpha\|_{H_3}/\delta\leq C/\delta$ for the solutions to the Euler-Voigt equations (the proof is similar to that of (7)). Inserting in relation (24) gives, after some manipulations,

$$\begin{aligned}\frac{d}{dt}(\|\omega\|_{H_3}^2+\alpha^2\|\nabla\omega\|_{H_3}^2) &\leq C(\|u_\delta^\alpha\|_{H_3}+\|u^{\alpha,\beta,\nu}\|_{H_3}+1)\|\omega\|_{H_3}^2+C\frac{\nu}{\delta^2}+\widehat{C}^2(T,\alpha,\delta,\nu) \\ &\leq C(\alpha,\beta,\delta,\nu)\big(\|\omega\|_{H_3}^2+\alpha^2\|\nabla\omega\|_{H_3}^2\big)+C\frac{\nu}{\delta^2}+\widehat{C}^2(T,\alpha,\delta,\nu).\end{aligned}$$

Where $C(\alpha,\beta,\delta,\nu):=\|u_\delta^\alpha\|_{H_3}+\|u^{\alpha,\beta,\nu}\|_{H_3}+1$ and $\widehat{C}(T,\alpha,\delta,\nu):=\big(\delta^{2-2s'}(\delta^2+C\alpha^2)(\|u_0^\beta-u_0\|_{H_3}^2+\|u_0^\beta\|_{H_3}^2)+\frac{\nu}{\delta^2}T\big)^{\frac{1}{2}}C^{\frac{1}{2}}(T)$. Then, using the Gronwall lemma, we deduce that

$$\|\omega\|_{L^\infty(0,T;H_3)}^2 \leq \Big(\big(\|\omega_0\|_{H_3}^2+\alpha^2\|\nabla\omega_0\|_{H_3}^2\big)+\big(C\frac{\nu}{\delta^2}+\widehat{C}^2(T,\alpha,\delta,\nu)\big)T\Big)e^{C(T,\alpha,\beta,\delta,\nu)},$$

with $C(T,\alpha,\beta,\delta,\nu):=C\int_0^T(\|u_\delta^\alpha(s)\|_{H_3}+\|u^{\alpha,\beta,\nu}(s)\|_{H_3}+1)ds$.

To conclude, we need a further estimate of $\alpha^2\|\nabla\omega_0\|_{H_3}^2=\alpha^2\|\nabla(u_{0,\delta}-u_0^\beta)\|_{H_3}^2$. Consider the regularized initial data $u_{0,\delta}^\beta$. It follows that

$$\begin{aligned}\alpha^2\|\nabla(u_{0,\delta}-u_0^\beta)\|_{H_3}^2 &\leq C\alpha^2\|u_{0,\delta}-u_{0,\delta}^\beta\|_{H_4}^2+C\alpha^2\|u_{0,\delta}^\beta-u_0^\beta\|_{H_4}^2 \\ &\leq C\frac{\alpha^2}{\delta^2}\|u_0^\beta-u_0\|_{H_3}^2+C\alpha^2\|u_0^\beta\|_{H_4}^2,\end{aligned} \tag{26}$$

and noting that $\|u_{0,\delta} - u_0^\beta\|_{H_3}^2 \leq C\|u_{0,\delta} - u_0\|_{H_3}^2 + C\|u_0^\beta - u_0\|_{H_3}^2$, we obtain

$$\begin{aligned}\|\omega\|^2_{L^\infty(0,T;H_3)} \leq & C\Big(\|u_{0,\delta} - u_0\|^2_{H_3} + C(1 + \frac{\alpha^2}{\delta^2})\|u_0^\beta - u_0\|^2_{H_3} \\ & + C\alpha^2\|u_0^\beta\|^2_{H_4} + \big(C\frac{\nu}{\delta^2} + \widehat{C}^2(T,\alpha,\delta,\nu)\big)T\Big)e^{C(T,\alpha,\beta,\delta,\nu)}.\end{aligned} \tag{27}$$

Estimate for $\|u_\delta^\alpha - u^{\alpha,\beta}\|_{L^\infty(0,T;H_3)}$: Taking the H_3-energy estimate for $\omega_\delta^{\alpha,\beta} := u_\delta^\alpha - u^{\alpha,\beta}$ (also in this case, we drop α, β and δ), we arrive at

$$\frac{d}{dt}(\|\omega\|^2_{H_3} + \alpha^2\|\nabla\omega\|^2_{H_3}) \leq (\|u_\delta^\alpha\|_{H_3} + \|u^{\alpha,\beta}\|_{H_3})\|\omega\|^2_{H_3} + \|u_\delta^\alpha\|_{H_4}\|\omega\|_{L^\infty}\|\omega\|_{H_3}.$$

We estimate the term $\|\omega\|_{L^\infty(\mathbb{T}^3)}$ in the usual way and we find

$$\begin{aligned}\frac{d}{dt}(\|\omega\|^2_{H_{s'}} + \alpha^2\|\nabla\omega\|^2_{H_{s'}}) &\leq (\|u_\delta^\alpha\|_{H_{s'+1}} + \|u^{\alpha,\beta}\|_{H_{s'+1}})\|\omega\|^2_{H_{s'}} \\ &\leq (\|u_\delta^\alpha\|_{H_{s'+1}} + \|u^{\alpha,\beta}\|_{H_{s'+1}})(\|\omega\|^2_{H_{s'}} + \alpha^2\|\nabla\omega\|^2_{H_{s'}}) \\ &\leq C(\|\omega\|^2_{H_{s'}} + \alpha^2\|\nabla\omega\|^2_{H_{s'}}).\end{aligned}$$

Then, applying the Gronwall lemma and using the bound (25), we get

$$\|\omega\|^2_{L^\infty(0,T;H_{s'})} \leq \big(\delta^{4-2s'}(\delta^2 + \alpha^2)(\|u_0^\beta - u_0\|^2_{H_3} + \|u_0^\beta\|^2_{H_3})\big)C(T).$$

making computations similar to those performed in (18)-(19), we finally obtain the estimate

$$\begin{aligned}\frac{d}{dt}(\|\omega\|^2_{H_3} + \alpha^2\|\nabla\omega\|^2_{H_3}) \leq & C(\|u_\delta^\alpha\|_{H_3} + \|u^{\alpha,\beta}\|_{H_3} + 1)(\|\omega\|^2_{H_3} + \alpha^2\|\nabla\omega\|^2_{H_3}) \\ & + \widehat{C}^2(T,\alpha,\beta,\delta),\end{aligned}$$

where $\widehat{C}(T,\alpha,\beta,\delta) := \big(\delta^{2-2s'}(\delta^2+\alpha^2)(\|u_0^\beta - u_0\|^2_{H_3} + \|u_0^\beta\|^2_{H_3})\big)^{\frac{1}{2}}C^{\frac{1}{2}}(T)$. Hence, we get

$$\begin{aligned}\|u_\delta^\alpha - u^{\alpha,\beta}\|^2_{L^\infty(0,T;H_3)} \leq & \big(\|u_{0,\delta} - u_0^\beta\|^2_{H_3} + \alpha^2\|\nabla(u_{0,\delta} - u_0^\beta)\|^2_{H_3} \\ & + \widehat{C}^2(T,\alpha,\beta,\delta)T\big)e^{C\int_0^T(\|u_\delta^\alpha(s)\|_{H_3} + \|u^{\alpha,\beta}(s)\|_{H_3}+1)ds}.\end{aligned}$$

Now, arguing as in (26), we can conclude that

$$\begin{aligned}\|u_\delta^\alpha - u^{\alpha,\beta}\|^2_{L^\infty(0,T;H_3)} \leq & C\Big(\|u_{0,\delta} - u_0\|^2_{H_3} + C(1 + \frac{\alpha^2}{\delta^2})\|u_0^\beta - u_0\|^2_{H_3} \\ & + \alpha^2\|u_0^\beta\|^2_{H_4} + \widehat{C}^2(T,\alpha,\beta,\delta)T\Big)e^{C(T,\alpha,\beta,\delta)},\end{aligned} \tag{28}$$

where $C(T,\alpha,\beta,\delta) := C\int_0^T(\|u_\delta^\alpha(s)\|_{H_3} + \|u^{\alpha,\beta}(s)\|_{H_3} + 1)ds$.

Estimate for $\|u^{\alpha,\beta} - u\|_{L^\infty(0,T;H_3)}$: Note that, up to a sub-sequence $\alpha_n \to 0$ as $n \to \infty$, the needed estimate on $\|u^{\alpha_n,\beta_n} - u\|_{L^\infty(0,T;H_3)}$ is provided by Theorem 3.3.

As a consequence of the above bound and the estimates (27) and (28), for any positive sequence $\{\nu_n\}$, with $\nu_n \to 0$ as $n \to \infty$, letting $\beta_n \to 0$, we can choose a pair of sequences $\alpha_n, \delta_n \to 0$ (look at the proof of Theorem 3.3), such that $\|u_{0,\delta_n} - u_0\|_{H_3}$, $\alpha_n^2\|u_0^{\beta_n}\|^2_{H_4}$, $\frac{\alpha_n}{\delta_n}$, $\frac{\alpha_n}{\delta_n^{s'-1}}$, and $\frac{\nu_n}{\delta_n^2}$ go to zero as n goes to infinity. Hence, we get

$$\|u^{\alpha_n,\beta_n,\nu_n} - u\|_{L^\infty(0,T;H_3)} \to 0 \quad \text{as } n \to \infty,$$

and the thesis follows. □

Remark 4.1. *The result holds also in spaces of more regular functions H_m, $m \geq 3$, by using essentially the same techniques.*

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